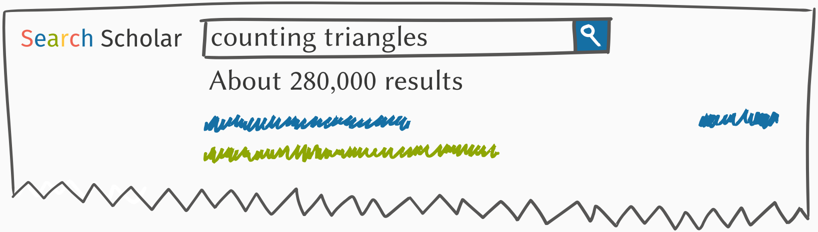
Sampling Arbitrary Subgraphs Exactly Uniformly in Sublinear Time

Hendrik Fichtenberger, Mingze Gao and Pan Peng

ICALP 2020

Subgraph Problems

subgraph problems are basic but popular:



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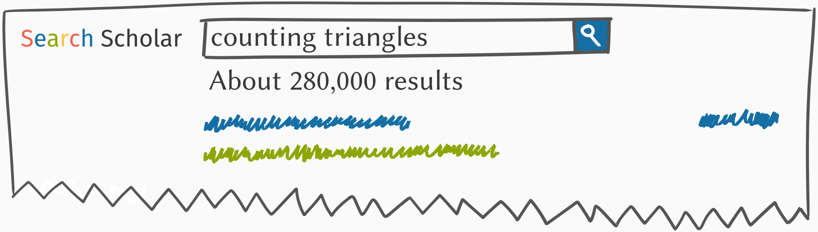

in this talk: sampling subgraphs in sublinear time

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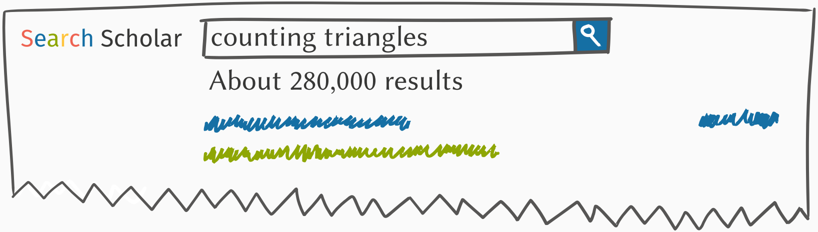

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↓

fixed subgraph
uniformly from
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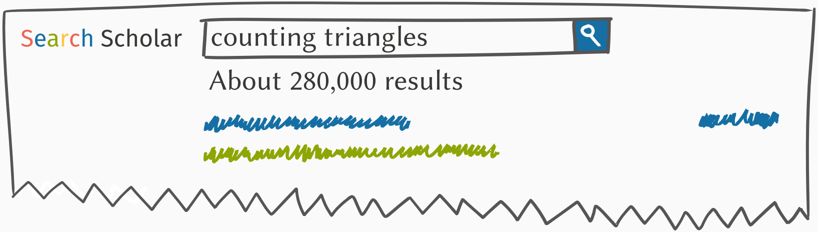
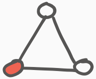


in this talk: sampling subgraphs in sublinear time

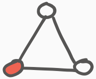
fixed subgraph uniformly from an input graph

need to define model of access to input graph



query for uniformly random vertex

Graph Models



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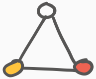

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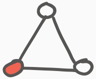

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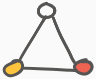

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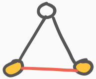

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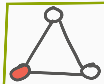

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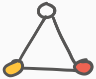

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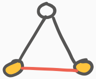

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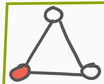

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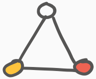

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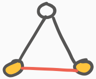

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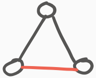

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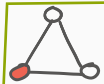

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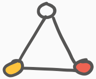

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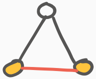

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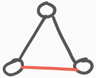

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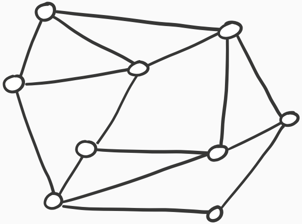
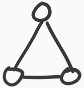
augmented general model

Simple Algorithm for the Augmented General Model



G



H

repeat until success:

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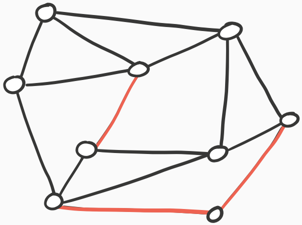

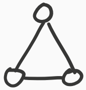

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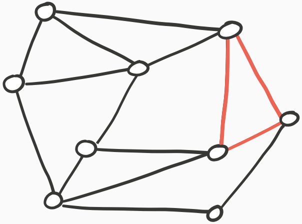
1. sample $|E(H)|$ edges uniformly at random
2. check wether they form a copy of H



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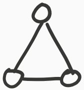

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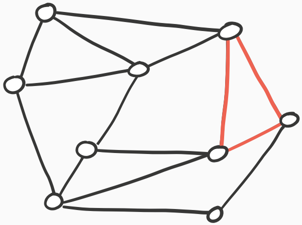

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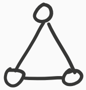

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- probability to sample a fixed copy of H: $\Theta\left(\frac{1}{m^{|E(H)|}}\right)$
- expected running time: $\Theta\left(\frac{m^{|E(H)|}}{\#H}\right)$

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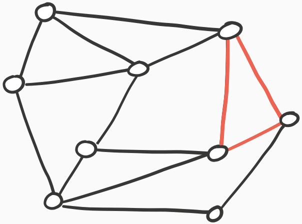

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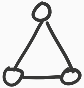

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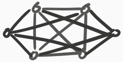
can we do better?

Related Work[1]

improvements over linear query bounds in $\tilde{O}(\ldots)$, all essentially tight for cliques

subgraph H	approximate counting	sampling *approx.* uniformly	sampling *exactly* uniformly
			
			
			

[1] Goldreich, Ron, RS&A’08; Eden, Rosenbaum, SOSA’18; Eden, Ron, Seshadhri, STOC’18; Assadi, Kapralov, Khanna, ITCS’18

Related Work[1]

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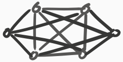
	$\frac{n}{\sqrt{m}}$ [GR08]	$\frac{n}{\sqrt{m}}$ [ER18]	
	$\frac{n}{(\#H)^{1/\lvert H\rvert}} + \frac{m^{\rho(H)}}{\#H}$ [ERS18]		
any			

general model

$\rho(H)$ is the frac. edge cover size of H, $|H|/2 \leq \rho(H) \leq |H|$

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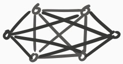	$\frac{n}{(\#H)^{1/\vert H\vert}} + \frac{m^{\rho(H)}}{\#H}$ [ERS18]		
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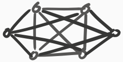	$\frac{n}{(\#H)^{1/\lvert H\rvert}} + \frac{m^{\rho(H)}}{\#H}$ [ERS18]	$\frac{m^{\rho(H)}}{\#H}$ [ERS18]	$\frac{m^{\rho(H)}}{\#H}$ [ERS18]
any	$\frac{m^{\rho(H)}}{\#H}$ [AKK18]	$\frac{m^{\rho(H)}}{\#H}$ [this work]	$\frac{m^{\rho(H)}}{\#H}$ [this work]

general model / augmented general model

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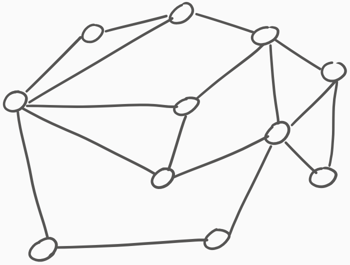
Our Results

Main Theorem

For any subgraph H, sampling *exactly uniformly* from all copies of H in an input graph G has expected query and time complexity $\mathcal{O}\left(\frac{m^{\rho(H)}}{\#H}\right)$ in the augmented general model.

This is essentially tight for cliques, even when we require only *almost* uniform sampling.

Fractional Edge Covers



Theorem [AKK18][2]

For every graph H, there is a minimum fractional edge cover by vertex-disjoint odd cycles and stars.

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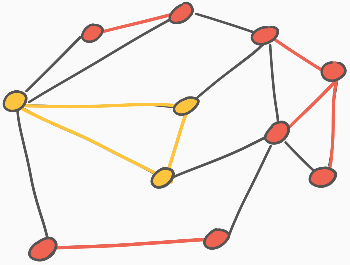

$\psi(H) = 6.5$

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The value $\psi_H(C)$ of a cover C is

$$\psi_H(C) = \sum_{\substack{k \in \{3,5,\ldots\} \\ C_k \in C}} \frac{k}{2} + \sum_{\substack{k \in \mathbb{N} \\ S_k \in C}} k.$$

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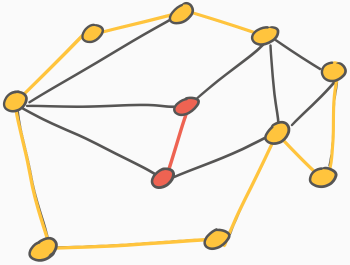

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We define $\rho(H) = \min_C \psi_H(C)$.

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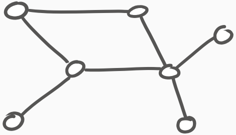
Sampling in the Augmented General Model

for subgraph H and input G:



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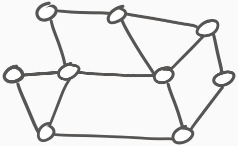

G

$\rho(H) = 5$

Sampling in the Augmented General Model

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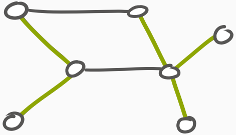

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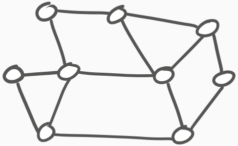

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Sampling in the Augmented General Model

for subgraph H and input G:

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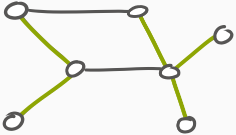
2. sample $\rho(H)$ edges from G



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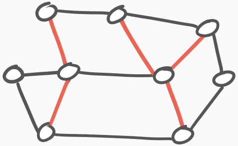

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Sampling in the Augmented General Model

for subgraph H and input G:

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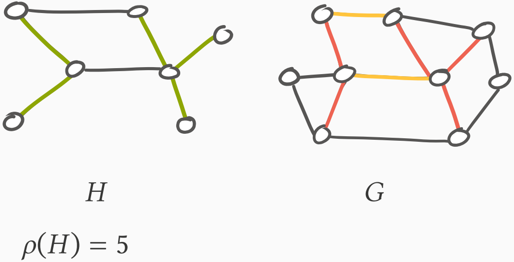

Sampling in the Augmented General Model

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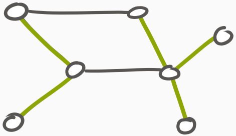
expected time to find it: $\Theta(m^{\rho(H)})$



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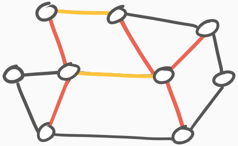
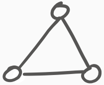


G

How About Odd Cycles?



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for odd cycles C_k: $\rho(C_k) = \frac{k}{2}$

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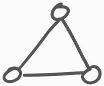

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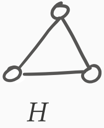

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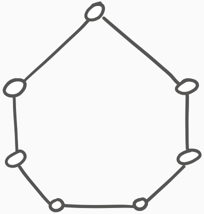

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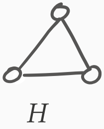

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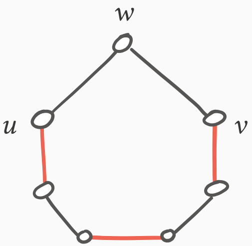
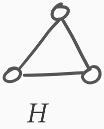


for odd cycles C_k:

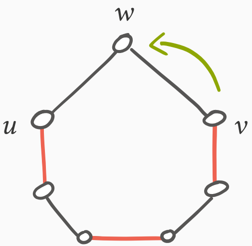
1. sample $\frac{k-1}{2}$ edges

How About Odd Cycles?



for odd cycles C_k: $\rho(C_k) = \frac{k}{2}$

how to sample $\frac{k}{2}$ edges?



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2. sample a common neighbor of u and v

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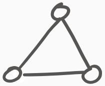

H

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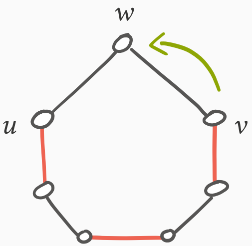

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neighbor query probability: $\Pr[w] = \frac{1}{d_v}$

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for all $w \in \Gamma(u) \cap \Gamma(v)$

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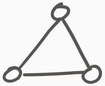

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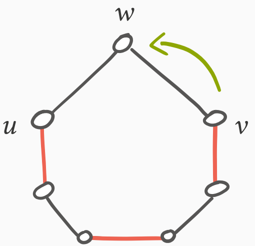

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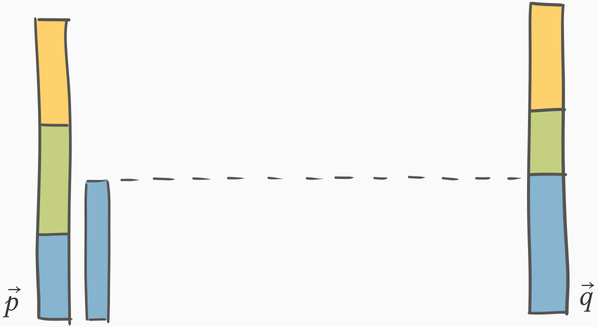
total probability to sample a fixed H: $\Theta\left(\frac{1}{m^{(k-1)/2}} \cdot \frac{1}{\sqrt{m}}\right) = \Theta\left(\frac{1}{m^{\rho(C_k)}}\right)$

Rejection Sampling

Problem

Given samples from distribution $\vec{p}$ on $[n]$, simulate sampling from $\vec{q}$.

Rejection Sampling



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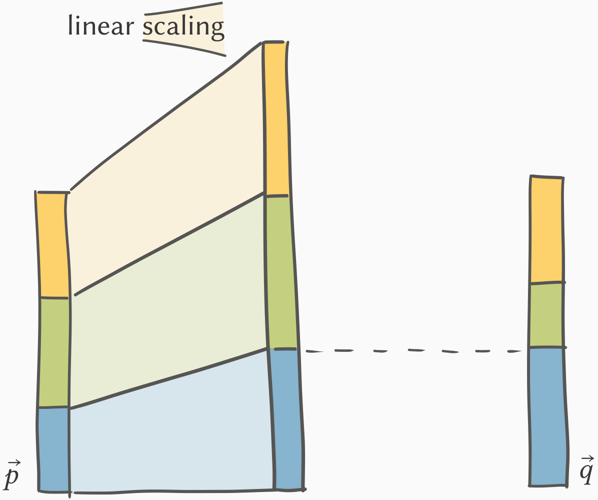

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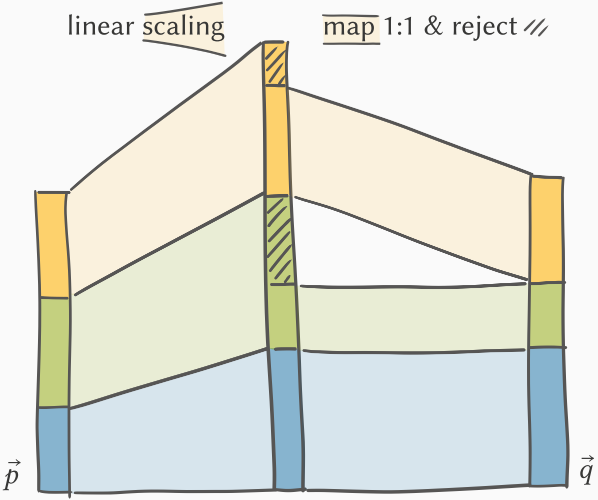
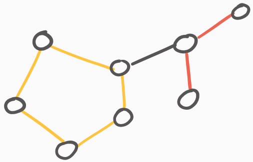


Problem

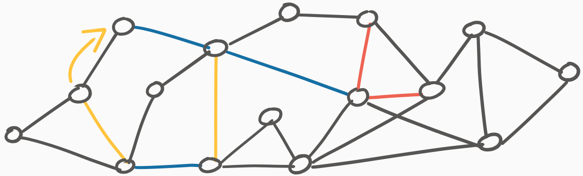
Given samples from distribution $\vec{p}$ on $[n]$, simulate sampling from $\vec{q}$.

1. scale $\vec{p}$ linearly by factor $s = \max_i \vec{q}(i)/\vec{p}(i)$
2. sample o from $\vec{p}$
3. sample x uniformly from $[0, 1]$
4. accept o if $x \leq \vec{q}(o)/(s\vec{p}(o))$, reject and repeat otherwise



H

1. decompose H into odd cycles C and stars S
2. repeat until success:
    a. sample edges from G as described
    b. check whether they form a copy of H using pair queries



G

sample H exactly uniformly in $\mathcal{O}\left(\frac{m^{\rho(H)}}{\#H}\right)$ expected time