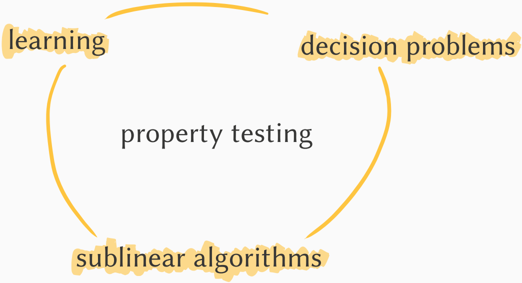
Property Testing of Graphs and the Role of Neighborhood Distributions

Hendrik Fichtenberger

February 11, 2020

Property Testing: in Context



Property Testing: in Context

property testing makes
approximative decisions

learning

decision problems

property testing

sublinear algorithms

Property Testing: in Context

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property testing minimizes
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Property Testing: in Context

proper PAC learning
implies property testing

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sublinear algorithms

property testing minimizes
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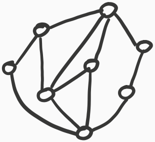
Property Testing: in a Nutshell



planar ✓

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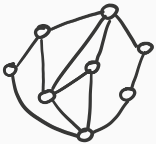

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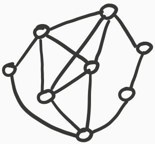

non-planar ✗

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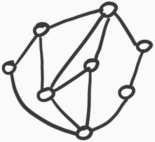

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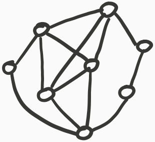

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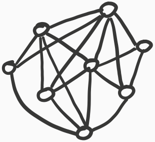

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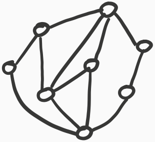

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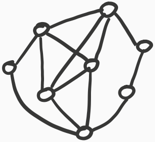

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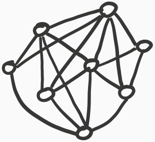

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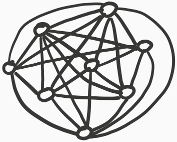

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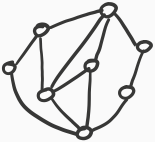

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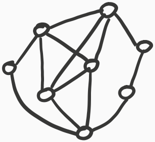

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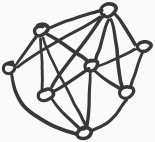

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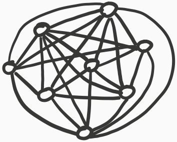
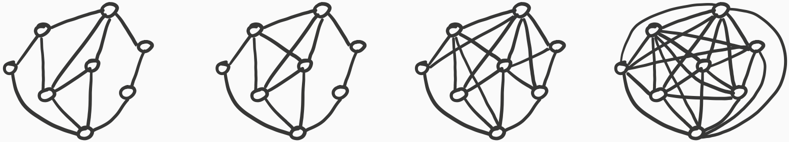


non-planar

time complexity: $\Omega(|V|)$

Property Testing: in a Nutshell



planar   non-planar   non-planar   non-planar

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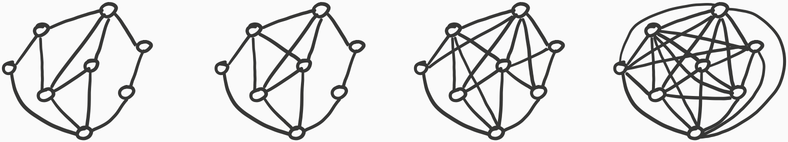

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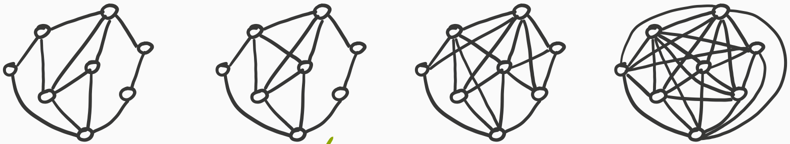

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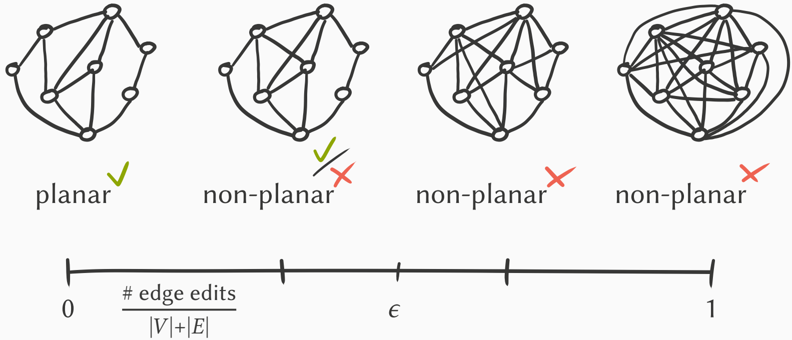

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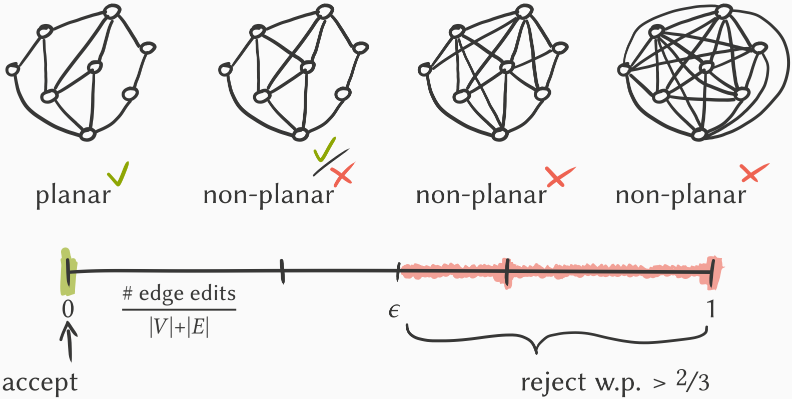

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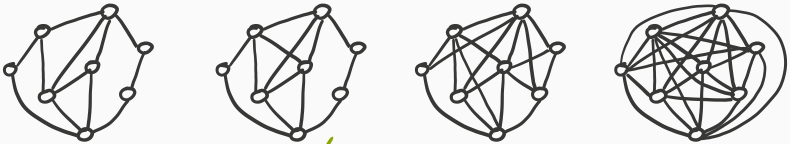

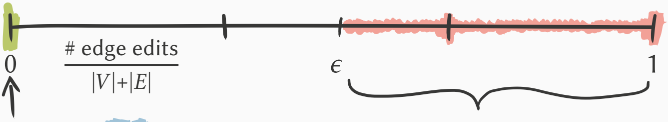

accept w.p. > 2/3

reject w.p. > 2/3

1-sided error / 2-sided error

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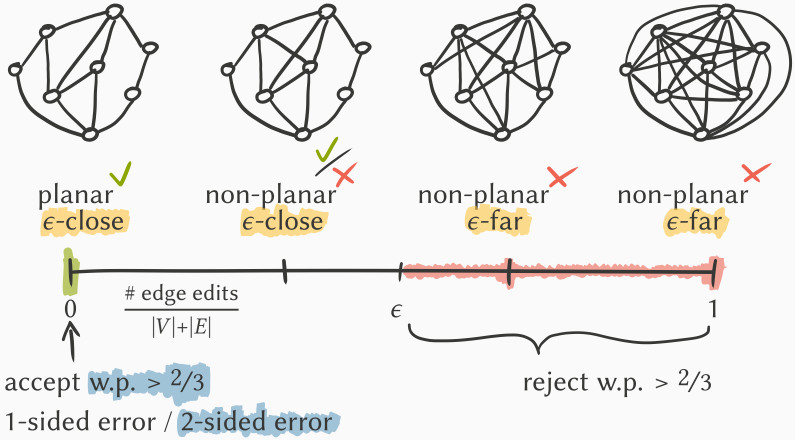

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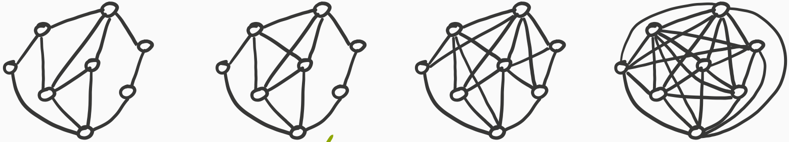

planar ϵ-close

non-planar ϵ-close

non-planar ϵ-far

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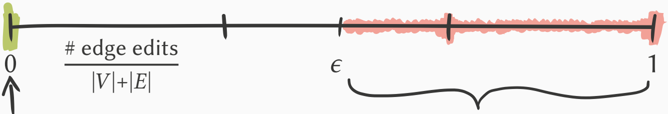

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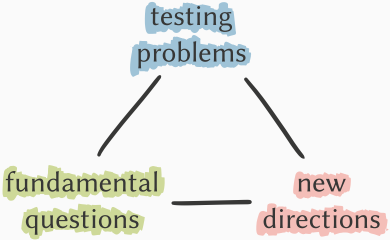
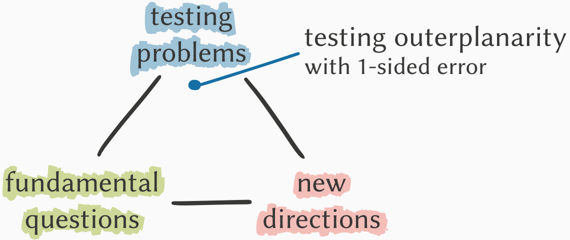
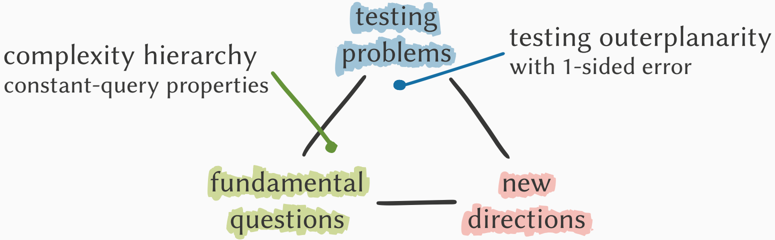
complexity: # queries to data structure


testing problems
fundamental questions
new directions

Property Testing: in my Thesis




testing
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complexity hierarchy
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Property Testing: in my Thesis

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distributed graph algorithms
testing expansion and conductance

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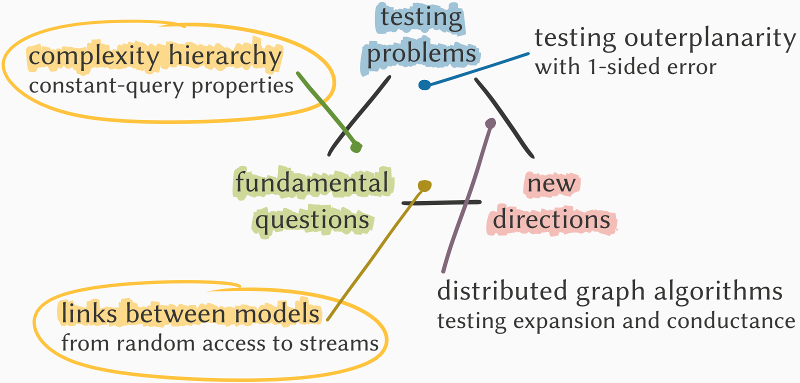

The Bounded-Degree Model

- ☒ bounded-degree model: $\forall v \in V : d(v) \leq d, d \in O(1), n := |V|$
- ☒ input structure: adjacency lists (1 query $\hat{=}$ 1 entry)
- ☒ error: 2-sided

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why?

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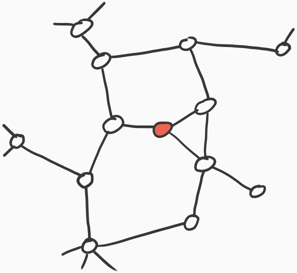
k-Disks / k-Hop Neighborhoods

$\text{disk}_k(v)$: subgraph induced by $\text{BFS}(v)$ of depth k



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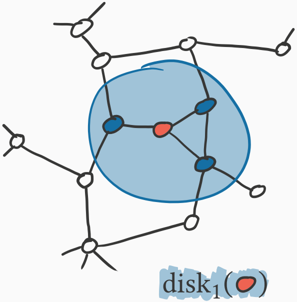

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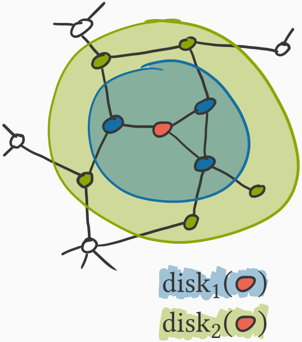

Goldreich, Ron, STOC'09

k-Disks / k-Hop Neighborhoods

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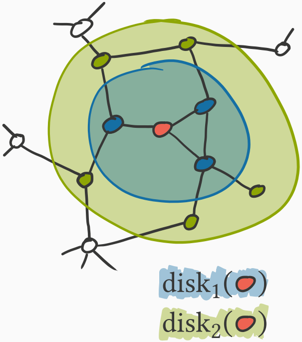
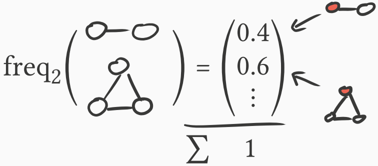


$\mathrm{freq}_k(G)$: for each k-disk isomorphism type calculate its share of vertices



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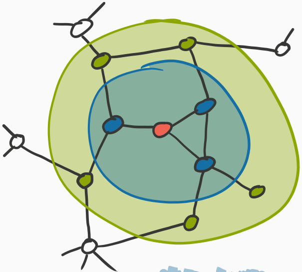

$\mathrm{disk}_1(\ )$

$\mathrm{disk}_2(\ )$

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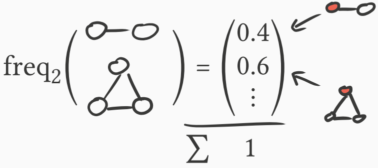
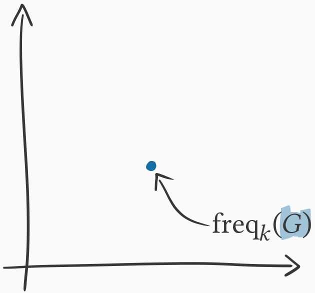
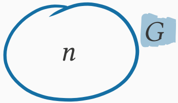


[GR'09]

Π constant-query testable iff $\mathrm{freq}_k(G)$ indicates membership

Small Frequency-Preserver Graphs





Theorem [Alon'11]

For every $\delta, k > 0$, there exists $M(\delta, k)$ such that for every G there exists H of size at most $M(\delta, k)$ and $\|\text{freq}_k(G) - \text{freq}_k(H)\|_1 < \delta$.

Alon, '11, see: Lovász, *Large Networks and Graph Limits*, Proposition 19.10

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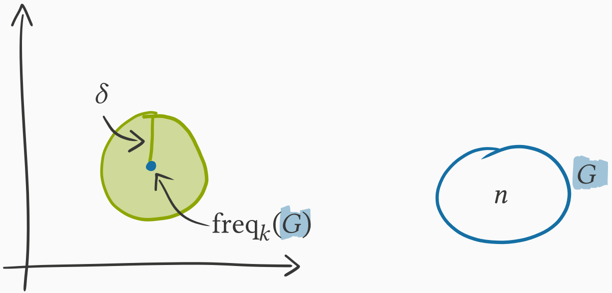

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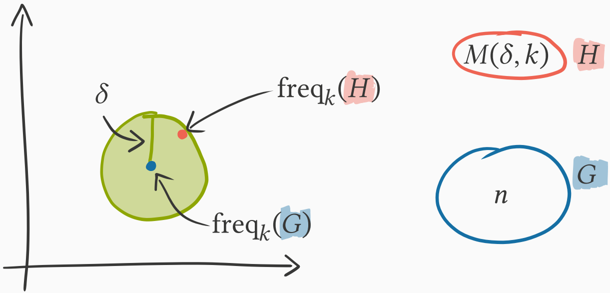

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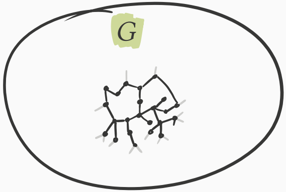
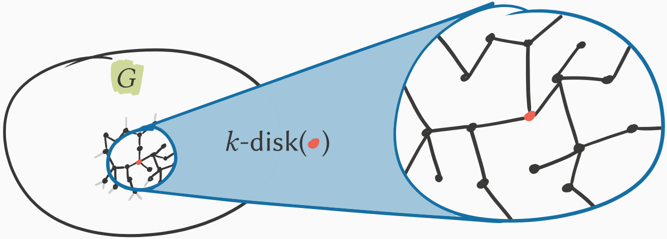


Theorem [with PS]

If all k-disks in G are cycle-free, there is a constant-time algorithm that constructs a frequency-preserver of size $M(\delta, k) \leq 10^4 d^{11k}/\epsilon^2\delta$.

Frequency-Preservers for Cycle-Free k-disks



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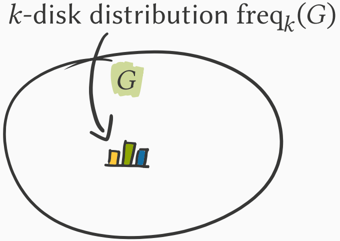

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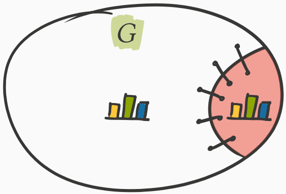

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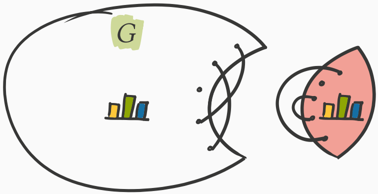
joint work with Peng, Sohler

Frequency-Preservers for Cycle-Free k-disks



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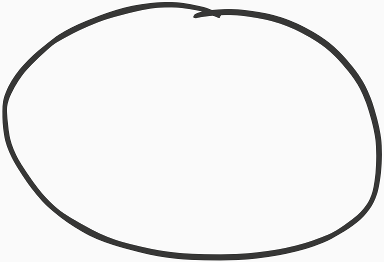
Hyperfinite Graphs



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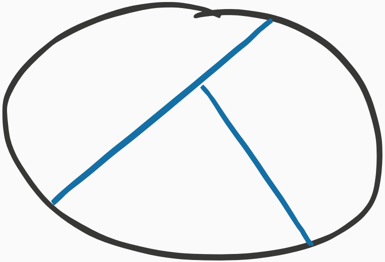

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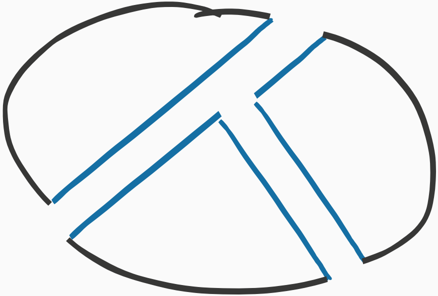

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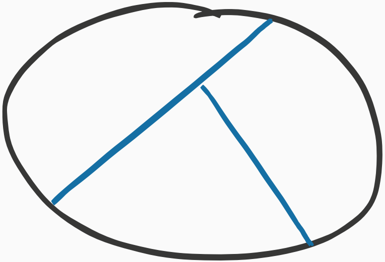

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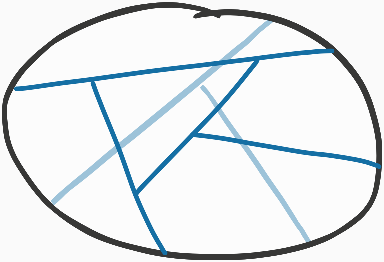

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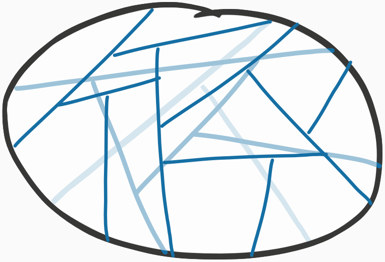

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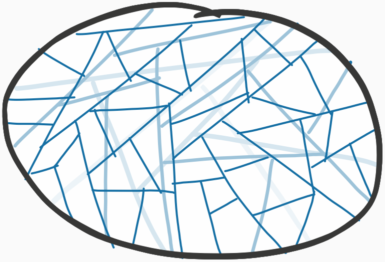

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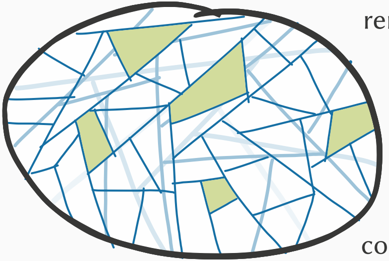
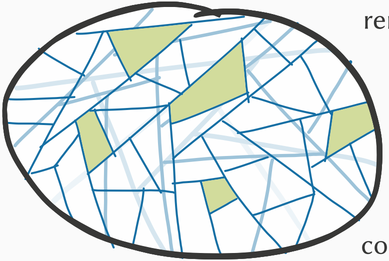


remove ϵdn edges for any $\epsilon > 0$

con. comp. of size $\leq \rho(\epsilon)$ for some ρ



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con. comp. of size $\leq \rho(\epsilon)$ for some ρ

Theorem (informal) [BSS'08]

If graph G is $\rho(\epsilon)$-hyperfinite, then any graph H with $\mathrm{freq}(G) \approx \mathrm{freq}(H)$ is $\rho'(\epsilon)$-hyperfinite for some $\rho' \approx \rho$.

Constant-Query Properties

Π is ρ-hyperfinite

Π has constant query complexity

Constant-Query Properties

[NS'11]

Π is ρ-hyperfinite

Π has constant query complexity

Constant-Query Properties

$$\begin{tikzcd}[column sep=huge] \Pi \text{ is } \rho\text{-hyperfinite} \arrow[r, bend left, "{[\text{NS'11}]}"] & \begin{array}{l}\Pi \text{ has constant} \\ \text{query complexity}\end{array} \arrow[l, bend left, "?"] \end{tikzcd}$$

Newman, Sohler, STOC '11

Constant-Query Properties

[NS'11]

Π is ρ-hyperfinite

Π has constant query complexity

e.g. Π = connectivity

contains expanders

Constant-Query Properties

[NS'11]

Π is ρ-hyperfinite

Π has constant query complexity

e.g. Π = connectivity

contains expanders

$\exists\, \Pi' \subseteq \Pi, |\Pi'| = \infty$:

Π' is ρ'-hyperfinite

Theorem [with PS]

Every non-trivial, constant-query testable property of bounded-degree graphs contains an infinite hyperfinite subproperty.

The General Graph Model

- ☒ bounded-degree model
- ☒ input structure: adjacency lists
- ☒ error: 2-sided

The General Graph Model

- ~~bounded degree model~~ general graphs
- input structure: adjacency lists
- error: 2-sided

The General Graph Model

- [x] ~~bounded degree model~~ general graphs
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The General Graph Model

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What can a constant-query property tester do?

The General Graph Model

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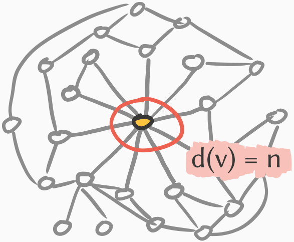

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BFS

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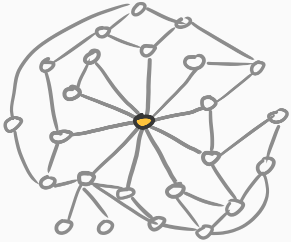

What can a constant-query property tester do?

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random / subsampling BFS

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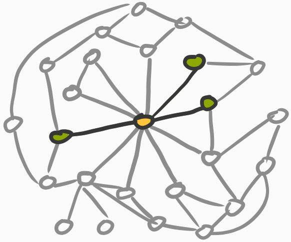

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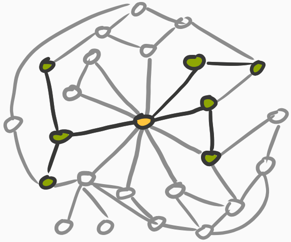

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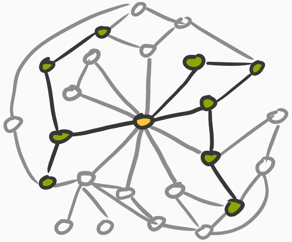

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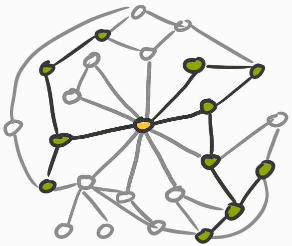

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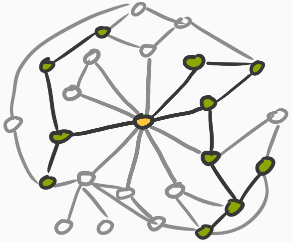

What can a constant-query property tester do?

~~BFS~~

random / subsampling BFS

Theorem (informal) [with CPS]

Every constant-query property tester for general graphs that queries adjacency lists can be reduced to (multiple) random BFS.

The Streaming Model

- ☒ general graphs
- ☒ input structure: adjacency lists
- ☒ error: 1-sided

The Streaming Model

- ☒ general graphs
- ☒ input structure: ~~adjacency lists~~ stream of edges
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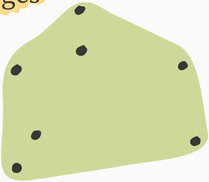
planar?



Monemizadeh, Muthukrishnan, Peng, Sohler, ICALP'17; Peng, Sohler, SODA'18

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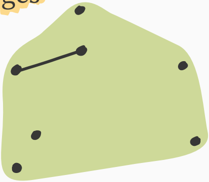

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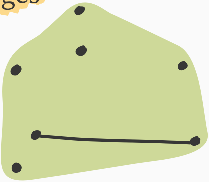

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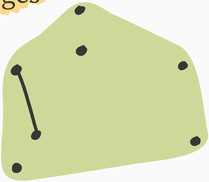

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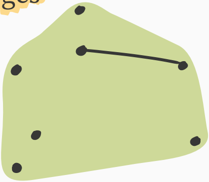

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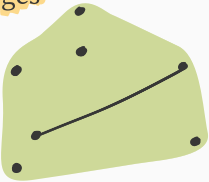

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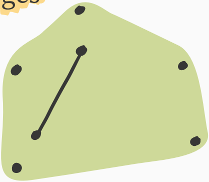

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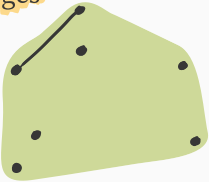

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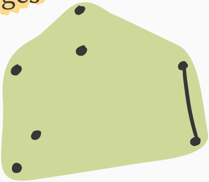

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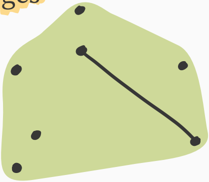

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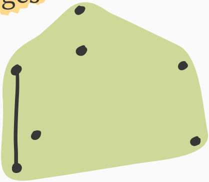

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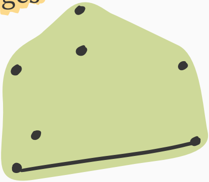

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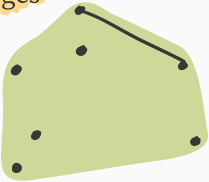

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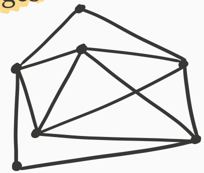

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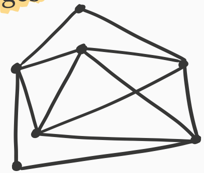

objective: $o(n)$ space

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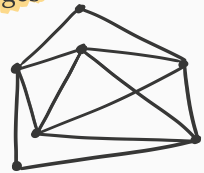

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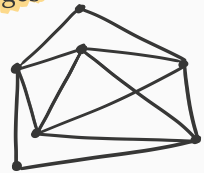

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Monemizadeh, Muthukrishnan, Peng, Sohler, ICALP'17; Peng, Sohler, SODA'18

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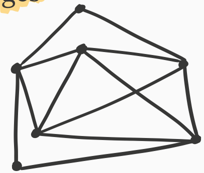

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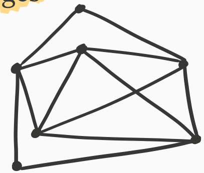

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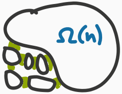
Theorem (informal) [with CPS]

One-sided error constant-query testers that query adjacency lists admit a $O(\log n)$-space random-order streaming tester.

Open Problems

- characterize constant-query properties
    - role of small connected components / cuts



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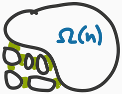
  - relate k-disk vectors and combinatorial properties



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    - degree / adjacency matrix queries

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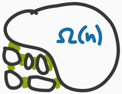

  - relate k-disk vectors and combinatorial properties

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  - 2-sided error