Every Testable ∞ Property of Bounded-Degree Graphs Contains an ∞ Hyperfinite ⊆-Property

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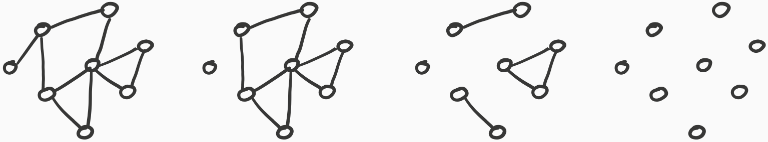
January 6, 2019

Property Testing in a Nutshell: Graph Connectivity



connected ✓ | disconnected ✗ | disconnected ✗ | disconnected ✗

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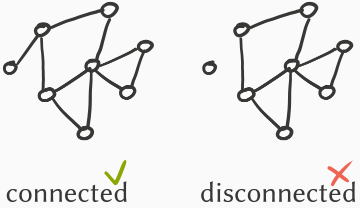

time complexity: $\Omega(|V| + |E|)$

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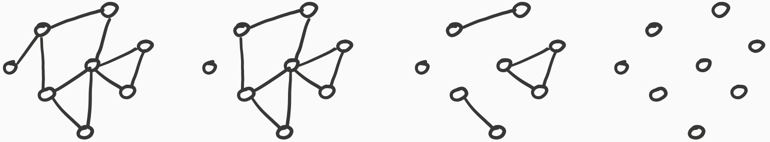

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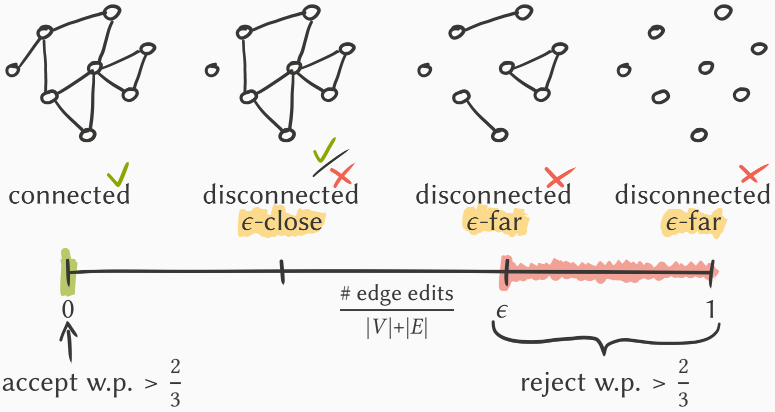

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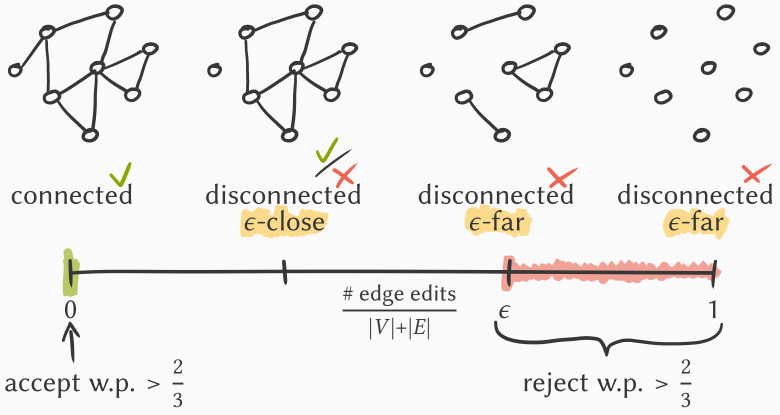

complexity: # queries to adjacency list entries

Property Testing of Bounded Degree Graphs

bounded degree graphs: $\forall v \in V \; : \; d(v) \leq d, d \in O(1)$

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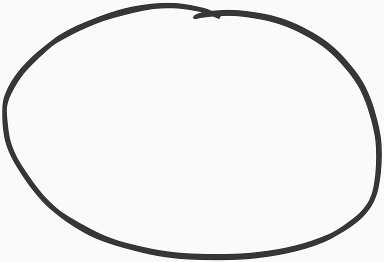
big picture?
little known about
$O(1)$-query testability

Hyperfinite Graphs

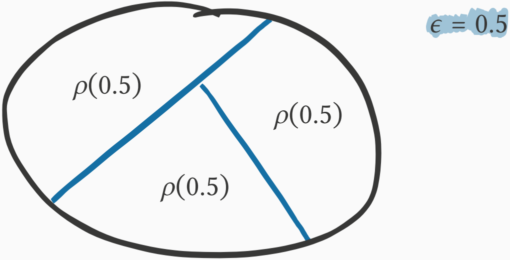


Definition

(ϵ, s)-hyperfinite: can remove at most $\epsilon d n$ edges to obtain connected components of size at most s

ρ-hyperfinite: $(\epsilon, \rho(\epsilon))$-hyperfinite for all $\epsilon \in (0, 1]$

Hyperfinite Graphs



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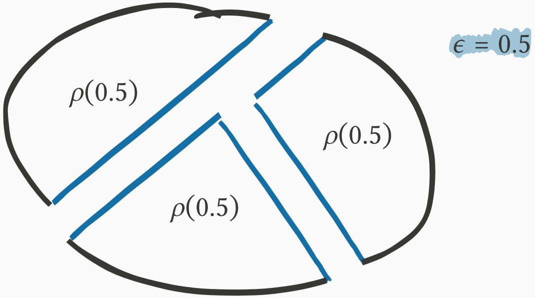

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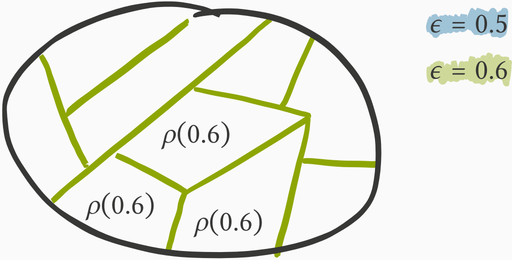

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Result

Let Π be a bounded-degree graph property

Π is ρ-hyperfinite

Π has constant query complexity

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[Newman, Sohler, 2011]

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e.g. Π = connectivity

contains expanders

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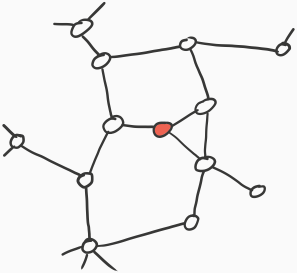
we show

$\exists\, \Pi' \subseteq \Pi$:

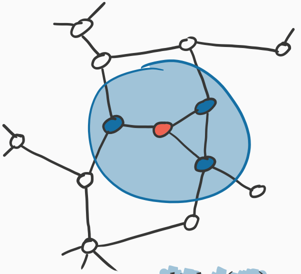
Π' is ρ'-hyperfinite

$\text{disk}_k(v)$: subgraph induced by BFS(v) of depth k



k-Disks and Frequency Vectors

$\mathrm{disk}_k(v)$: subgraph induced by BFS(v) of depth k



$\mathrm{disk}_1(\,)$

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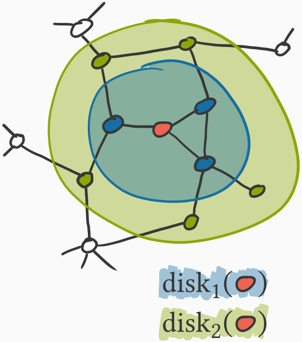

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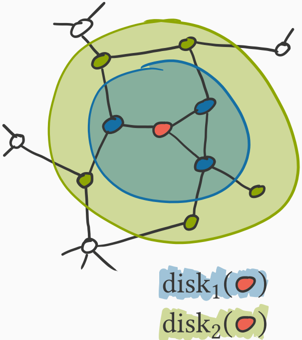
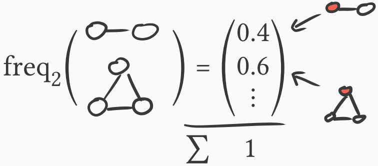


$\mathrm{freq}_k(G)$: for each k-disk isomorphism type calculate its share of vertices



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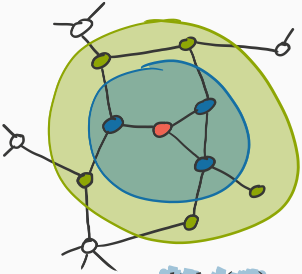
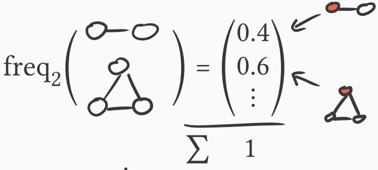


$\text{disk}_1(\,)$

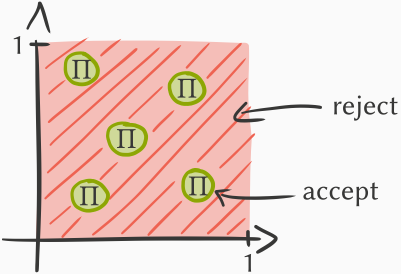
$\text{disk}_2(\,)$

$\text{freq}_k(G)$: for each k-disk isomorphism type calculate its share of vertices



frequency vector is a locality feature

One Thing That is Known

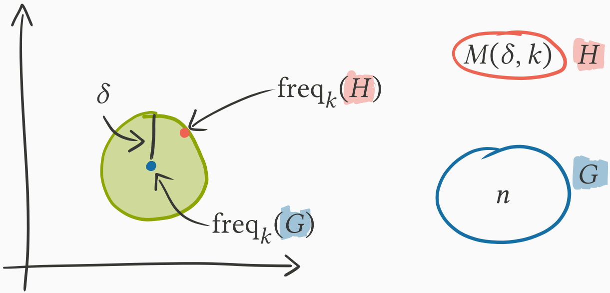


Lemma [Goldreich, Ron, 2009]

Every property tester with query complexity $q(\epsilon)$ can be transformed into an algorithm that

1. computes an approximation $\widetilde{\mathrm{freq}}_{cq(\epsilon)}(G)$ of $\mathrm{freq}_{cq(\epsilon)}(G)$
2. accepts iff $\|\widetilde{\mathrm{freq}}_{cq(\epsilon)}(G) - \mathrm{freq}_{cq(\epsilon)}(G')\|_1 \le \frac{1}{c' q(\epsilon)}$ for any $G' \in \Pi$

Small Frequency-Preserver Graphs



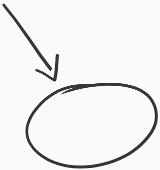
Lemma [Alon, 2010]

For every $\delta, k > 0$, there exists $M(\delta, k)$ such that for every G there exists H of size at most $M(\delta, k)$ and $\|\mathrm{freq}_k(G) - \mathrm{freq}_k(H)\|_1 < \delta$.

Transformation Step

Sketch: complement $\overline{\Pi}$ of Π contains hyperfinite subproperty

start w/ G ϵ-far from Π



rel. $\overline{\Pi}$	ϵ-far from Π
size	n
hypf.	?

freq. v. change	original

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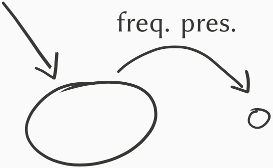
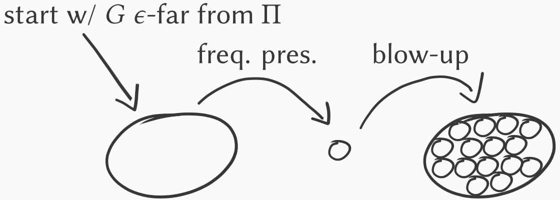


rel. $\overline{\Pi}$	ϵ-far from Π	?
size	n	$M(\delta, k)$
hypf.	?	?

freq. v. original $< \delta$
change

Sketch: complement $\overline{\Pi}$ of Π contains hyperfinite subproperty

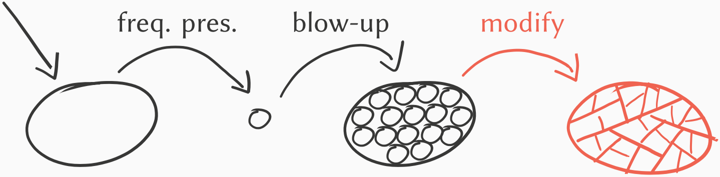


rel. $\overline{\Pi}$	ϵ-far from Π	?	ϵ/c-far from Π
size	n	$M(\delta, k)$	n
hypf.	?	?	$(0, M(\delta, k))$-hyperfinite
freq. v. change	original	$< \delta$	$= 0$

Transformation Step

Sketch: ~~complement $\overline{\Pi}$ of~~ Π contains hyperfinite subproperty

start w/ $G \in \Pi$



rel. Π	in Π	?	ϵ-close to Π	in Π
size	n	$M(\delta, k)$	n	n
hypf.	?	?	$(0, M(\delta, k))$-hyperfinite	$(\epsilon, M(\delta, k))$-hyperfinite
freq. v. change	original	$< \delta$	$= 0$	$< \delta$

Summary

Let Π be (inf.) constant-query testable property of bounded degree graphs:

1. Π contains (inf.) hyperfinite subproperty
2. $\overline{\Pi}$ contains (inf.) hyperfinite subproperty
3. all $G \in \Pi$ may be far from expander (even if Π is not hyperfinite)
4. partitioning theorem that preserves frequency vector of non-expaning subparts